Subdiffusive behavior generated by irrational rotations

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Abstract

The origin of deterministic diffusion is a matter of discussion. We study the asymptotic distributions of the sums $y_n(x) = \sum_{k=0}^{n-1} \psi(x + k\alpha)$, where ψ is a periodic function of bounded variation and α an irrational number. It is known that no diffusion process will be observed. Nevertheless, we find a picewise constant function ψ and an increasing sequence of integer $(n_j)_j$ such that the limit distribution of the sequence $(y_{n_j}/\sqrt{j})_j$ is Gaussian (with stricly positive variance). If α is of constant type, we show that the sequence $(n_j)_j$ may be taken to grow exponentially (this is close to optimal in some sense, and one has $||y_{n_j}||_{\mathrm{L}^2} \sim \max_{0\leq k\leq n_j} ||y_k||_{\mathrm{L}^2}$ as $j \to \infty$). We give an heuristic link with the theory of expanding maps of the interval.

1 Introduction

Some purely deterministic dynamical systems may generate diffusion process. Such a diffusion is always due to uncertainty on initial conditions. If a distribution is initialy concentrated in one point, it will remain so under the flow of a deterministic system. But if the initial conditions are distributed on some larger set of the phase space, it may well be that the distribution evolves diffusively.

Some cases of deterministic diffusion have been succesfully investigated. Let us mention the theory of expanding maps of the interval [3], and the important result by Bunimovich and Sinai about the Lorentz gas [2]. Nevertheless, there is an open debate about the origin of diffusion. In the two previous examples, the underlying dynamical system is hyperbolic ; and it has been suggested that macroscopic diffusion is generally due to microscopic chaos [6]. But numerical experiments with systems of zero Lyapunov exponents show that diffusion may happen even in the absence of hyperbolicity [4].

The rotation of the circle by an irrational angle is a well known example of ergodic non hyperbolic dynamical system. We will show that a diffusive behavior may be generated with this system, if we

*Partially supported by the Belgian IAP program P6/02 and by the University of Helsinki.

restrict the times we oberve it to an apropriate subsequence. One will see (section 2) that the system under consideration behaves like an hyperbolic one, when restricting our attention to some special subsequences.

This supports our personal view that hyperbolicity is, roughly speaking, the basic ingredient of deterministic diffusion. Nevertheless, it is mathematically too precise and too strong to be the good one in general. Indeed, if we look at a system at large scale, one does not need exponential separation of trajectories at infinitesimal level, but only at some smaller scale. This paper is intended to illustrate this point of view in a rather extreem case.

Let $\mathbf{T} = \mathbf{R}/\mathbf{Z}$. If $u \in \mathrm{L}^1(\mathbf{T},\mathbf{R})$, one defines $\mathrm{Var}(u) = \sup\{\int_0^1 uv'\mathrm{d}x : v \in \mathrm{C}^1(\mathbf{T},\mathbf{R}), ||v||_{\mathrm{L}^\infty} \leq 1\}$. One defines also the set $\mathrm{BV}(\mathbf{T},\mathbf{R}) = \{u \in \mathrm{L}^1(\mathbf{T},\mathbf{R}) : \mathrm{Var}(u) < \infty\}$.

Let $\psi \in \mathrm{BV}(\mathbf{T},\mathbf{R})$ be such that $\int_0^1 \psi \mathrm{d}x = 0$. Let $\alpha \in \mathbf{R} - \mathbf{Q}$. We consider the map

$$\mathrm{F} : \mathbf{T} \times \mathbf{R} \to \mathbf{T} \times \mathbf{R} : (x, y) \mapsto (x + \alpha, y + \psi(x)). \tag{1}$$

If $n \in \mathbf{N}$, one writes $\mathrm{F}^n(x, y) = (x + n\alpha, y + y_n(x))$. Explicitly, one has $y_n(x) = \sum_{k=0}^{n-1} \psi(x + k\alpha)$ for $n \geq 1$. Altough y_n depends on ψ and α, one will not generaly write it. Let m_{L} be the Lebesgue measure on $\mathbf{T}$. The space $(\mathbf{T}, m_{\mathrm{L}})$ is then a probability space, and $(y_n)_{n\geq 0} \subset \mathrm{BV}(\mathbf{T},\mathbf{R})$ is a sequence of random variables on this space.

The sequence $(y_n)_{n\geq 0}$ has been widely studied [1][5][8][10]. Here are two important informations. First, the sequence $(y_n)_{n\geq 0}$ is bounded in $\mathrm{L}^2(\mathbf{T},\mathbf{R})$ if and only if there exists $u \in \mathrm{L}^2(\mathbf{T},\mathbf{R})$ such that $\mathrm{R}_\alpha u - u = \psi$ (where by definition $\mathrm{R}_\alpha u(x) = u(x + \alpha)$) ([8] p.183). Next, let p/q be an irreductible fraction such that $|\alpha - p/q| \leq 1/q^2$ (by Dirichlet theorem, there are infinitely many such fractions). Denjoy-Koksma inequality asserts that $||y_q||_{\mathrm{L}^\infty} \leq \mathrm{Var}(\psi)$ ([8] p.73).

So, let $\psi \in \mathrm{BV}(\mathbf{T},\mathbf{R})$ be such that the equation $\mathrm{R}_\alpha u - u = \psi$ has no solution in $\mathrm{L}^2(\mathbf{T},\mathbf{R})$. Can we find an increasing sequence $(n_j)_{\geq 1} \subset \mathbf{N}$ such that $y_{n_j}/\sqrt{j}$ should be asymptotically normally distributed (with stricly positive variance) ? Proposition 1 answers this question positively. This means that, if we looked at the system at the times n_j only, we should observe a diffusion process. But how fast has to grow the sequence $(n_j)_{j\geq 1}$? If α is of a particular type (see later), we will see in proposition 2 that it may be taken to grow exponentially, and that it may not grow much slower.

However, from a mathematical point of view, those questions are not the most natural ones. For $j \geq 0$, let $m_j \in \mathbf{N}$ be such that $||y_{m_j}||_{\mathrm{L}^2} = \max_{0\leq m\leq j} ||y_m||_{\mathrm{L}^2}$ (and take the smallest one if there are more than one possibility). What can be said about the asymptotic distribution of the sequence $(y_{m_j})_{j\geq 0} \triangleq (z_j)_{j\geq 0}$? Proposition 2 leads us to think that, if the number α is of constant type (see later), and if the sequence is adequately rescaled, its distribution is asymptotically normal. However, because it is actually concerned with another sequence than $(z_j)_{j\geq 0}$, it does not allow us to claim that.

We define the function ψ_* by

$$\psi_*(x) = 1 \quad \text{if} \quad 0 \leq x < 1/2, \qquad \psi_*(x) = -1 \quad \text{if} \quad 1/2 \leq x < 1. \tag{2}$$

It is known that there is no $u \in \mathrm{L}^2(\mathbf{T}, \mathbf{R})$ that solves the equation $\mathrm{R}_\alpha u - u = \psi_*$ (lemma 2, section 2). Let $g(\sigma)$ be the probability measure on $\mathbf{R}$ that admits the density $f(x) = e^{-x^2/2\sigma^2}/\sqrt{2\pi\sigma^2}$ ($\sigma > 0$).

Proposition 1 *If $\psi = \psi_*$ and if $\alpha \in \mathbf{R} - \mathbf{Q}$, there exists an increasing sequence $(n_j)_{j\geq 1} \subset \mathbf{N}$ such that $y_{n_j}/\sqrt{j} \xrightarrow{\mathrm{D}} g(1)$ as $j \to \infty$.*

This proposition is quite vague, because the sequence $(n_j)_{j\geq 1}$ is completely unknown. Nevertheless, we believe it has some interest. First, the result is valid for any irrational number α. Next, the proof is not technical but contains the principal ideas we need for proving our second result. Finally, it allows us to make a clear heuristic link between our case and the theory of expanding maps of the interval (see section 2, after lemma 4).

Nevertheless, one is interested in finding a sequence $(n_j)_{j\geq 1}$ in proposition 1 that grows as slow as possible with j. Let $(p_n/q_n)_{n\geq 0} \subset \mathbf{Q}$ be the convergents of α, and let $(a_n)_{n\geq 0} \subset \mathbf{N}$ be its partial quotients (see [7] for definitions) (one will usually not write explicitly the dependence of p_n/q_n and a_n on α). The proof of proposition 1 shows that $n_j = q_1 + \cdots + q_j$ is maybe a good condidate. However, for some numbers α, the sequence $(q_n)_{n\geq 0}$ grows quite fast with n (superexponentially), and it should then not be clear at all whether this choice is optimal.

So, let us introduce a particular class of numbers. One says that $\alpha \in \mathbf{R} - \mathbf{Q}$ is of *constant type* if there exists $\mathrm{C}(\alpha) > 0$ such that, for every $q \in \mathbf{Z}_0$ and for every $p \in \mathbf{Z}$, $|q\alpha - p| \geq \mathrm{C}(\alpha)/|q|$. Equivalently, α is of constant type if its sequence $(a_n)_{n\geq 0}$ of partial quotients is bounded. This implies that the sequence $(q_n)_{n\geq 0}$ grows only exponentially with n. Those numbers form a set of zero Lebesgue measure.

We note, however, that the sequence $n_j = q_1 + \cdots + q_j$ may sometimes be just a wrong choice. For example, if α is the golden number, then $(q_n)_{n\geq 0}$ is the Fibonacci sequence, and one has $q_1 + \cdots + q_j = q_{j+2} - 2$. Therefore, by Denjoy-Koksma inequality, one should have $||y_{n_j}/\sqrt{j}||_{\mathrm{L}^\infty} \to 0$ as $j \to \infty$.

If $u : \mathbf{T} \mapsto \mathbf{R}$ is measurable, one defines its distribution μ_f as follows : μ_f is the measure on $\mathbf{R}$ such that, for every $a \in \mathbf{R}$, $\mu_f((-\infty, a]) = m_{\mathrm{L}}(\{x \in \mathbf{T} : u(x) \leq a\})$. For $\mathrm{A}, d \geq 1$, one defines also the set

$$\mathrm{E}(\mathrm{A}, d) = \{\alpha \in \mathbf{R} - \mathbf{Q} : \forall n \geq 1, \mathrm{A} \leq a_n \leq d\mathrm{A}\}. \tag{3}$$

Proposition 2 *Let $\psi = \psi_*$. Let $\mathrm{A}, d \geq 1$, and let $\alpha \in \mathrm{E}(\mathrm{A}, d)$. Let $n \geq 1$. Let $r_n = q_1 + \cdots + q_n$. Let $\sigma_n = ||y_{r_n}/\sqrt{n}||_{\mathrm{L}^2}$. Let μ_n be the distribution of f_{r_n}. If A is large enough, then there exists $\mathrm{C}, \epsilon > 0$ such that $\mathrm{C} \geq \sigma_n \geq \epsilon$. Moreover, $\mu_n - g(\sigma_n) \rightharpoonup 0$ as $n \to \infty$.*

Let $p > 0$, and let $(n_j)_{j\geq 1} \subset \mathbf{N}$ be such that $n_j \leq \mathrm{C} j^p$ for some $\mathrm{C} > 0$. It follows from lemma 9 (section 4) that $||y_{n_j}/\sqrt{j}||_{\mathrm{L}^2} \to 0$ as $j \to \infty$. This justifies the claim that n_j may not grow much slower that exponentially for proposition 1 to be true. On another hand, one may ask what information proposition 2 gives us about the sequence $(z_j)_{j\geq 0}$. One is acually only able to establish some heuristic,

namely that the sequences $(y_{r_n})_{n\geq 1}$ and $(z_{r_n})_{n\geq 1}$ grow at the same rate:

$$\epsilon\sqrt{n} \leq ||y_{r_n}||_{\mathrm{L}^2} \leq ||z_{r_n}||_{\mathrm{L}^2} \leq \mathrm{C}\sqrt{n}, \tag{4}$$

where the last inequality is obtained by lemma 9.

We are left with at least two technical questions. First, what happens when $\psi \neq \psi_*$? Like the second one, this question has not been investigated for proposition 1, because the goal there was mostly to give an example. In proposition 2, the choice $\psi = \psi_*$ is only needed to prove $\sigma_n \geq \epsilon$ (lemma 10, section 4). It follows from the proof of this lemma that other choices should be possible.

Second, what happens when the Lebesgue measure on $\mathbf{T}$ is replaced by another probability measure μ ? Propostion 3 (section 3) is valid for an arbitrary probability measure μ. If μ admits a density of bounded variation, lemmas 8 and 9 (section 4) should still be valid. If, moreover, this density is bounded from bellow by a strictly positive number, lemma 10 (section 4) should be valid, and therefore proposition 2 should hold.

The organisation of the paper is as follows. Proposition 1 is shown in section 2. In section 3, one shows an abstract central limit theorem ; this section is independent of the others. One proves proposition 2 in section 4.

The letter C is used to denote a strictly positive constant that may vary from place to place.

2 Proof of Proposition 1

Let $\alpha \in \mathbf{R}-\mathbf{Q}$. Let $(p_n/q_k)_{k\geq 0}$ be its convergents, and $(a_k)_{k\geq 0}$ its partial quotients. Let $\psi \in \mathrm{BV}(\mathbf{T}, \mathbf{R})$ be such that $\int_0^1 \psi \mathrm{d}x = 0$.

Lemma 1 *Let $n \geq 0$.*

1) *Of the fractions p_n/q_n et p_{n+1}/q_{n+1}, one at least satisfies $|\alpha - p/q| < 1/2q^2$.*

2) *If q_n is even, then q_{n+1} is odd.*

3) *If q_n and q_{n+2} are even, then $|\alpha - p_{n+1}/q_{n+1}| < 1/2q_{n+1}^2$.*

4) *From four consecutive convergents, one at least has an odd denominator and satisfies the inequality $|\alpha - p/q| < 1/2q^2$.*

Proof. For 1), see [7] p.152. Let us show 2) by contradiction. Let us suppose we have found a smallest $j \in \mathbf{N}$ such that q_j and q_{j+1} are even. We have $j \geq 1$ and therefore $q_{j+1} = a_{j+1}q_j + q_{j-1}$. Because q_{j-1} is odd and q_j even, q_{j+1} should also be odd. Let us show 3). By 2), q_{n+1} is odd, and on the other hand we have that $q_{n+2} = a_{n+2}q_{n+1} + q_n$. The number a_{n+2} has to be even, and therefore $a_{n+2} \geq 2$. The result follows from the inequality $|\alpha - p_{n+1}/q_{n+1}| < 1/a_{n+2}q_{n+1}^2$. Finally, 4) is obtained by considering all the possibilities. $\square$

If $u \in \mathrm{L}^2(\mathbf{T},\mathbf{R})$, if $k \in \mathbf{Z}$, one writes $\hat{u}(k) = \int_0^1 u(x)e^{-2i\pi kx}\mathrm{d}x$. If $u \in \mathrm{BV}(\mathbf{T},\mathbf{R})$, one has $|\hat{u}(k)| \leq \mathrm{Var}(u)/2\pi|k|$ for $k \neq 0$. One has

$$\widehat{y_n}(k) = \frac{1-e^{2i\pi nk\alpha}}{1-e^{2i\pi k\alpha}}\hat{\psi}(k), \quad (n \geq 1, k \in \mathbf{Z}_0). \tag{5}$$

Let us also introduce the following notation : if $x \in \mathbf{R}$, one writes $|x|_{\mathbf{T}} = \inf_{p\in\mathbf{Z}} |x-p|$. One checks that $4|x|_{\mathbf{T}} \leq |1-e^{2i\pi x}| \leq 2\pi|x|_{\mathbf{T}}$, and that $\forall x \in \mathbf{R}$, $\forall m \in \mathbf{Z}$, $|1-e^{2i\pi mx}| \leq |m|.|1-e^{2i\pi x}|$.

Lemma 2 *Let $\psi = \psi_*$. There exists no $u \in \mathrm{L}^2(\mathbf{T},\mathbf{R})$ such that $\mathrm{R}_\alpha u - u = \psi_*$.*

Proof. A solution u should be such that $\hat{u}(k) = \hat{\psi}_*(k)/(e^{2i\pi k\alpha}-1) = -2i/\pi k(e^{2i\pi k\alpha}-1)$ if k is odd. By lemma 1, there exist infinitely many odd k such that $|k\alpha|_{\mathbf{T}} < 1/|k|$, and therefore $\hat{u}(k)$ should not goes to 0 as $k \to \infty$. $\square$

Lemma 3 *If $n \to \infty$, $y_{q_n} \rightharpoonup 0$ in $\mathrm{L}^2(\mathbf{T},\mathbf{R})$.*

Proof. By Denjoy-Koksma inequality, $||y_{q_n}||_{\mathrm{L}^\infty} \leq \mathrm{Var}(\psi)$. Therefore, we only need to check that, if $k \in \mathbf{Z}_0$, $\widehat{y_{q_n}}(k) \to 0$ as $k \to \infty$. By (5), $|\widehat{y_{q_n}}(k)| \leq \frac{\mathrm{Var}(\psi)}{2\pi|k|}|1-e^{2i\pi q_n k\alpha}|$ if $k \neq 0$. But

$$|1-e^{2i\pi q_n k\alpha}| \leq |k|.|1-e^{2i\pi q_n\alpha}| \leq 2\pi|k|.|q_n\alpha|_{\mathbf{T}} \to 0 \quad \text{as} \quad n \to \infty. \quad \square$$

Following [8] p.64, we then give some informations about some finite sequences $(\overline{n\alpha})_n$. If $p/q \in \mathbf{Q}$ is irreducible, one has $\{\overline{j.p/q}\}_{0\leq j\leq q-1} = \{j/q\}_{0\leq j\leq q-1}$. We say that $p/q \in \mathbf{Q}$ (p/q irreductible) is a rational approximation of α for the constant $0 < \beta \leq 1$ if the inequality $|\alpha - p/q| < \beta/q^2$ is satisfied. Let us write $\{\overline{j\alpha}\}_{0\leq j\leq q-1} = \{\alpha_j\}_{0\leq j\leq q-1}$, where $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_{q-1} < 1$. If $\alpha > p/q$, one has $k\alpha - k.p/q < k\beta/q^2 < 1/q$ if $1 \leq k \leq q-1$, and one writes

$$0 = \alpha_0 < \frac{1}{q} < \alpha_1 < \frac{2}{q} < \alpha_2 < \cdots < \frac{q-1}{q} < \alpha_{q-1} < 1.$$

If $\alpha < p/q$, one has then

$$\alpha_0 = 0 < \alpha_1 < \frac{1}{q} < \alpha_2 < \frac{2}{q} < \cdots < \alpha_{q-1} < \frac{q-1}{q} < 1.$$

In both cases one has $|\alpha_j - j/q| < \beta/q$ $(1 \leq j \leq q-1)$. The following lemma gives a slight improvement of Denjoy-Koksma inequality when $\psi = \psi_*$.

Lemma 4 *Let $\psi = \psi_*$. Let p/q be a rational approximation of α for the constant $\beta \leq 1/2$, and suppose that q is odd. One has $||y_q||_{\mathrm{L}^\infty} \leq 1$.*

Proof. One has $\phi = \sum_{k=0}^{q-1} \mathrm{R}_{kp/q}\psi = \sum_{k=0}^{q-1} \mathrm{R}_{k/q}\psi$. One has $\phi(x) = \psi(qx)$. Indeed, one has $\mathrm{R}_{1/q}\phi = \phi$ and $\phi|_{[0,1/q[} = (q-1)/2 + \mathrm{R}_{(q-1)/2}\psi|_{[0,1/q[} - (q-1)/2$. Let us then write $\{\overline{j\alpha}\}_{0\leq j\leq q-1} = \{\alpha_j\}_{0\leq j\leq q-1}$,

where $0 = \alpha_0 < \alpha_1 < \cdots < \alpha_{q-1} < 1$. One has $y_q = \sum_{k=0}^{q-1} \mathrm{R}_{k\alpha}\psi = \sum_{k=1}^{q-1}(\mathrm{R}_{\alpha_k} - \mathrm{R}_{k/q})\psi + \phi$. If $\alpha > p/q$, one has

$$(\mathrm{R}_{\alpha_k} - \mathrm{R}_{k/q})\psi(x) = \begin{cases} +2 & \text{if} \quad x \in [1-\alpha_k, 1-k/q[, \\ -2 & \text{if} \quad x \in [1/2-\alpha_k, 1/2-k/q[\quad (\mathrm{mod}1), \\ 0 & \text{otherwise,} \end{cases}$$

and if $\alpha < p/q$, one has

$$(\mathrm{R}_{\alpha_k} - \mathrm{R}_{k/q})\psi(x) = \begin{cases} -2 & \text{if} \quad x \in [1-k/q, 1-\alpha_k[, \\ +2 & \text{if} \quad x \in [1/2-k/q, 1/2-\alpha_k[\quad (\mathrm{mod}1), \\ 0 & \text{otherwise.} \end{cases}$$

To fix the ideas, let us condsider the case $\alpha > p/q$. One has

$$y_q\big|_{[j/q,(j+1)/q[} = 2\chi_{[\frac{j+1}{q}-\delta_1(j),\frac{j+1}{q}[} - 2\chi_{[\frac{j}{q}+\frac{1}{2q}-\delta_2(j),\frac{j}{q}+\frac{1}{2q}[} + \chi_{[\frac{j}{q},\frac{j}{q}+\frac{1}{2q}[} - \chi_{[\frac{j}{q}+\frac{1}{2q},\frac{j+1}{q}[},$$

where $0 \le \delta_1(j), \delta_2(j) < 1/2q$. $\square$

We can now give an heuristic explanation of proposition 1. A map $\mathrm{T} : \mathbf{T} \to \mathbf{T}$ is called *expanding*, if it is indefinitely differentiable except on a finite number of points, and if there exists $\rho > 1$ such that $\mathrm{T}' \ge \rho$ whenever T' is definied.

Let $(\rho_k)_{k\ge1} \subset]1,\infty[$ be such that $\rho_k \to \infty$ as $k \to \infty$. Let also $(\gamma_k)_{k\ge1} \subset \mathbf{R}$. If n is odd, y_n takes only odd values. Therefore, if p/q is a rational approximation of α for a constant $\beta \le 1/2$, and if q is odd, y_q takes only the values ±1 (by lemma 4). Then, by lemmas 1 and 3, there exists a subsequence $(\tilde{p}_k/\tilde{q}_k)_{k\ge1} \subset (p_n/q_n)_{n\ge0}$ such that, for every $k \ge 1$, and for every number γ_k, one may write $\mathrm{R}_{\gamma_k} y_{\tilde{q}_k} = \psi_\star \circ \mathrm{T}_k$, where T_k is an expanding map such that $\mathrm{T}_k' \ge \rho_k$ almost everywhere (for example, one may take T_k picewise linear).

For $k \ge 1$, let $n_k = \tilde{q}_1 + \cdots + \tilde{q}_k$ and define $f_1 = y_{\tilde{q}_1}$ and $f_k = \mathrm{R}_{n_{k-1}\alpha} y_{\tilde{q}_k}$. One has $y_{n_k} = \sum_{j=1}^k f_j$. Therefore, in view of [3], one suspects $y_{n_k}/\sqrt{k}$ to be asymptotically normaly distributed if ρ_k grows fast enough with k (at least exponentially). The proof we will now give of proposition 1 is greatly simplified by the fact that we allow ρ_k to grow as fast as we want with k. In the two next section, we prove basically that an exponential grow of ρ_k is enough.

We now come to the proof of proposition 1 (we retake the notations of the heuristic explanation). One has $f_n = \pm1$ and therefore $f_n^2 = 1$. Let $(\delta_k)_{k\ge1} \subset \mathbf{R}_0^+$ be such that $\sum_{j=1}^k \delta_j/\sqrt{k} \to 0$ as $k \to \infty$. Let $\beta \in [-1,1]$. One may suppose that, for every $k \ge 1$,

$$|\int_0^1 f_{n_{k+1}} e^{i\beta(f_{n_1}+\cdots+f_{n_k})} \mathrm{d}x| \le \delta_k.$$

Indeed, for some $m(k) \in \mathbf{N}$, one may write $[0,1] = \bigcup_{j=1}^{m(k)} \mathrm{I}_j$, in such a way that $e^{i\beta(f_{n_1}+\cdots+f_{n_k})}$ is constant on each I_j $(1 \le j \le m(k))$. But, by lemma 3, one may suppose that $|\int_{\mathrm{I}_j} f_{n_{k+1}} \mathrm{d}x| \le \delta_k/m(k)$ $(1 \le j \le m(k))$.

Let $\lambda \in \mathbf{R}$. For $k \geq 1$ big enough, one has $|\lambda/\sqrt{k}| \leq 1$ and $|1 - \frac{\lambda^2}{2k}| \leq 1$. Therefore, using the fact that, for $k \geq 2$, $e^{i\lambda(f_{n_1}+\cdots+f_{n_k})/\sqrt{k}} = e^{i\lambda(f_{n_1}+\cdots+f_{n_{k-1}})/\sqrt{k}}(1 + i\lambda f_{n_k}/\sqrt{k} - \lambda^2/2k + O|\lambda|^3 f_{n_k}^3/k^3)$, one obtains

$$|\int_0^1 e^{i\frac{\lambda}{\sqrt{k}}(f_{n_1}+\cdots+f_{n_k})}\mathrm{d}x - \prod_{j=1}^{k}(1 - \frac{\lambda^2}{2k})| \leq \frac{|\lambda|}{\sqrt{k}}\sum_{j=1}^{k}\delta_j + \frac{\lambda^2}{2k}\sum_{j=1}^{k}\delta_j + O\frac{|\lambda|^3}{\sqrt{k}}. \quad \square$$

3 Central Limit Theorem

Let μ be a probability measure on $\mathbf{T}$. In this section $\mathrm{L}^p(\mathbf{T},\mathbf{R}) = \mathrm{L}^p(\mathbf{T},\mathbf{R},\mathrm{d}\mu)$ ($p \geq 1$) (but the definition of $\mathrm{BV}(\mathbf{T},\mathbf{R})$ is not affected by the choice of the measure μ).

Proposition 3 *Let $(q_k)_{k\geq 1} \subset \mathbf{N}_0$, and suppose there exists $\mathrm{A} > 1$ such that $q_{k+1} \geq \mathrm{A}q_k$ for $k \geq 1$. Let $(f_k)_{k\geq 1} \subset \mathrm{BV}(\mathbf{T},\mathbf{R})$ be a sequence of random variables on $(\mathbf{T},\mu)$ such that $\int_0^1 f_k \mathrm{d}\mu = 0$ ($k \geq 1$). Let $\mathrm{S}_n = f_1 + \cdots + f_n$. Suppose that there exists a number C such that*

1) *for every $k \geq 1$, $||f_k||_{\mathrm{L}^\infty} \leq \mathrm{C}$ and $\mathrm{Var}(f_k) \leq \mathrm{C}q_k$,*

2) *for some $\beta \in \mathbf{R}$, for every $\phi \in \mathrm{BV}(\mathbf{T},\mathbf{R})$ such that $\int_0^1 \phi \mathrm{d}\mu = 0$, and for every $t \geq s \geq 1$,*

$$|\int_0^1 \phi f_s \mathrm{d}\mu| \leq \mathrm{C}.\mathrm{Var}(\phi)\frac{s^\beta}{q_s}, \qquad |\int_0^1 \phi f_s f_t \mathrm{d}\mu| \leq \mathrm{C}.\mathrm{Var}(\phi)\frac{s^\beta}{q_s}. \tag{6}$$

3) *Suppose also there exists $\sigma > 0$ such that $||\mathrm{S}_n/\sqrt{n}||_{\mathrm{L}^2} \to \sigma$ as $n \to \infty$.*

Then, $\mathrm{S}_n/\sqrt{n} \xrightarrow{\mathrm{D}} g(\sigma)$ as $n \to \infty$.

Proof. We begin by a

Lemma 5 *Under the hypothesis of proposition 3, there exists a number C such that for every $m, n \geq 1$, $||\sum_{k=m}^{m+n} f_k||_{\mathrm{L}^4} \leq \mathrm{C}(n\ln(n+m))^{1/2}$.*

Proof. One has

$$\begin{aligned} ||\sum_{k=m}^{m+n} f_k||_{\mathrm{L}^4}^4 &= \sum_{m\leq s,t,u,v\leq m+n}\int_0^1 f_s f_t f_u f_v \mathrm{d}\mu \leq \mathrm{C}\sum_{m\leq s\leq t\leq u\leq v\leq m+n}|\int_0^1 f_s f_t f_u f_v \mathrm{d}\mu| \\ &= \mathrm{C}\sum_{m\leq t\leq u\leq m+n}\mathrm{S}(t,u) \end{aligned}$$

with

$$\mathrm{S}(t,u) = \begin{array}{ccc} |\int_0^1 f_m f_t f_u f_u \mathrm{d}\mu| & +\cdots+ & |\int_0^1 f_t f_t f_u f_u \mathrm{d}\mu| + \\ \vdots & \ddots & \vdots \\ |\int_0^1 f_m f_t f_u f_{m+n} \mathrm{d}\mu| & +\cdots+ & |\int_0^1 f_t f_t f_u f_{m+n} \mathrm{d}\mu|. \end{array} \tag{7}$$

Untill the end of this proof, one assumes $m \leq s \leq t \leq u \leq v \leq m+n$.

If $f, g \in \mathrm{BV}(\mathbf{T}, \mathbf{R})$, one has $\mathrm{Var}(fg) \le ||f||_{\mathrm{L}^\infty}\mathrm{Var}(g) + ||g||_{\mathrm{L}^\infty}\mathrm{Var}(f)$. Therefore, using all the hypothesises exept 3), one founds that there exists $\rho > 0$ such that

$$\begin{aligned} |\int_0^1 f_s f_t f_u f_v \mathrm{d}\mu| &\le \mathrm{C}\frac{q_u}{q_v} v^\beta \le \mathrm{C}e^{-2\rho(v-u)}v^\beta, \\ |\int_0^1 f_s f_t f_u f_v \mathrm{d}\mu| &\le \mathrm{C}(\frac{q_t}{q_u}u^\beta + \frac{q_s}{q_t}t^\beta) \le \mathrm{C}(e^{-2\rho(u-t)}u^\beta + e^{-2\rho(t-s)}t^\beta). \end{aligned}$$

Therefore

$$\begin{aligned} |\textstyle\int_0^1 f_s f_t f_u f_v \mathrm{d}\mu| &\le \mathrm{C}e^{-\rho(v-u)} && \text{if} \quad v-u \ge (\beta/\rho)\ln(m+n), \\ |\textstyle\int_0^1 f_s f_t f_u f_v \mathrm{d}\mu| &\le \mathrm{C}(e^{-\rho(u-t)} + e^{-\rho(t-s)}) && \text{if} \quad u-t \ge (\beta/\rho)\ln(m+n), t-s \ge (\beta/\rho)\ln(m+n). \end{aligned}$$

We now estimate $\mathrm{S}(t,u)$ for fixed t, u. First we consider the case where $u - t < (\beta/\rho)\ln(m+n)$. Each line in (7) is estimated by $n.\gamma$, where $\gamma = \mathrm{C}$ if $v - u < (\beta/\rho)\ln(m+n)$ and $\gamma = \mathrm{C}e^{-\rho(v-u)}$ otherwise. Therefore, one gets

$$\mathrm{S}(t,u) \le \mathrm{C}n\Big((\beta/\rho)\ln(m+n) + \sum_{k\ge1} e^{-\rho k}\Big) \le \mathrm{C}n\ln(m+n). \tag{8}$$

Next, we consider the case where $u - t \ge (\beta/\rho)\ln(m+n)$. We write the decomposition $\mathrm{S}(t,u) = \mathrm{S}_1(t,u) + \mathrm{S}_2(t,u) + \mathrm{S}_3(t,u)$. The sum $\mathrm{S}_1(t,u)$ is taken over the terms in (7) for which $v - u \ge u - t$ (lower lines in (7)). One has

$$\mathrm{S}_1(t,u) \le \mathrm{C}n\sum_{k\ge1} e^{-\rho(u-t+k)} \le \mathrm{C}ne^{-\rho(u-t)}. \tag{9}$$

The sum $\mathrm{S}_2(t,u)$ is taken over the terms in (7) that are not in $\mathrm{S}_1(t,u)$ and for which $t - s \ge u - t$ (left columns without their lower parts in (7)). One has

$$\mathrm{S}_2(t,u) \le \mathrm{C}n(u-t)e^{-\rho(u-t)}. \tag{10}$$

The sum $\mathrm{S}_3(t,u)$ is taken over the remaining terms in (7) (upper right corner in (7)). For less than $\mathrm{C}\ln^2(m+n)$ terms, one has $v - u < (\beta/\rho)\ln(m+n)$ and $t - s < (\beta/\rho)\ln(m+n)$. Therefore, one has

$$\mathrm{S}_3(t,u) \le \mathrm{C}(\ln^2(m+n) + \sum_{k\ge1} ke^{-\rho(k-1)}) \le \mathrm{C}\ln^2(m+n). \tag{11}$$

By (8 - 11), one has

$$\begin{aligned} ||\sum_{k=m}^{m+n} f_k||_{\mathrm{L}^4}^4 \quad \le \quad & \mathrm{C} \sum_{\substack{m\le t\le u\le m+n \\ u-t<(\beta/\rho)\ln(m+n)}} n\ln(m+n) + \\ & \mathrm{C} \sum_{m\le t\le u\le m+n} (ne^{-\rho(u-t)} + n(u-t)e^{-\rho(u-t)} + \ln^2(m+n)) \le \mathrm{C}n^2\ln^2(m+n). \quad \square \end{aligned}$$

If $n \ge 1$, set $n_1 = \lfloor n^{3/4} \rfloor$ and $n_2 = \lfloor n^{1/5} \rfloor$. In the sequel, we suppose that n is large enough to have $n_2 \ge 1$. One writes $\mathrm{S}_n = \sum_{k=1}^{p(n)}(\mathrm{X}_{nk} + \mathrm{Y}_{nk})$ where $p(n)$ is the smallest integer such that

$p(n).(n_1+n_2) \geq n$ and where

$$\begin{aligned} \mathrm{X}_{nk} &= f_{(k-1)(n_1+n_2)+1} + \cdots + f_{(k-1)(n_1+n_2)+n_1}, \\ \mathrm{Y}_{nk} &= f_{(k-1)(n_1+n_2)+n_1+1} + \cdots + f_{k(n_1+n_2)} \end{aligned}$$

$(1 \leq k \leq p(n)-1$; for $k = p(n)$ the definition is the same but one puts 0 instead of f_j whenever $j > n)$. We have that $p(n)/n^{1/4} \to 1$ as $n \to \infty$.

For $\lambda \in \mathbf{R}$, and $1 \leq k \leq p(n)$, we define $\mathrm{I}_{nk}(\lambda) = \int_0^1 e^{i\frac{\lambda}{\sqrt{n}}(\mathrm{X}_{n1}+\cdots+\mathrm{X}_{nk})}\mathrm{d}\mu$; we put also $\mathrm{I}_{n0}(\lambda) = 1$.

Lemma 6 *Under the hypothesis of proposition 3, if $\lambda \in \mathbf{R}$ and if $1 \leq k \leq p(n)$, one has*

$$\mathrm{I}_{nk}(\lambda) = (1 - \frac{\lambda^2}{2n}\int_0^1 \mathrm{X}_{nk}^2 \mathrm{d}\mu)\mathrm{I}_{n(k-1)}(\lambda) + r_{nk}(\lambda)$$

with $|r_{nk}(\lambda)| \leq \mathrm{C}(\lambda^2+|\lambda|^3)n_2^{5\beta}\mathrm{A}^{-n_2} + \mathrm{C}|\lambda|^3(\frac{1}{n^{1/4}})^{3/2}\ln^{3/2} n$.

Proof. Let us only consider the most difficult case $k \geq 2$. One has

$$\mathrm{I}_{nk}(\lambda) = \int_0^1 \Big(1 + i\frac{\lambda}{\sqrt{n}}\mathrm{X}_{nk} - \frac{\lambda^2}{2n}\mathrm{X}_{nk}^2 - \frac{i}{2}\int_0^{\frac{\lambda}{\sqrt{n}}\mathrm{X}_{nk}} e^{it}(\frac{\lambda}{\sqrt{n}}\mathrm{X}_{nk} - t)^2 \mathrm{d}t\Big)e^{i\frac{\lambda}{\sqrt{n}}(\mathrm{X}_{n1}+\cdots+\mathrm{X}_{n(k-1)})}\mathrm{d}\mu.$$

On the one hand, there exists a number C such that $q_1 + \cdots + q_n \leq \mathrm{C}q_n$, and so

$$\mathrm{Var}(e^{i\frac{\lambda}{\sqrt{n}}(\mathrm{X}_{n1}+\cdots+\mathrm{X}_{n(k-1)})}) \leq \mathrm{C}\frac{\lambda}{\sqrt{n}}q_{(k-2)(n_1+n_2)+n_1}.$$

Therefore,

$$\begin{aligned} |\int_0^1 \frac{i\lambda}{\sqrt{n}}\mathrm{X}_{nk}e^{i\frac{\lambda}{\sqrt{n}}(\mathrm{X}_{n1}+\cdots+\mathrm{X}_{n(k-1)})}\mathrm{d}\mu| &\leq \frac{\lambda}{\sqrt{n}}\sum_{j=1}^{n_1}|\int_0^1 f_{(k-1)(n_1+n_2)+j}e^{i\frac{\lambda}{\sqrt{n}}(\mathrm{X}_{n1}+\cdots+\mathrm{X}_{n(k-1)})}\mathrm{d}\mu| \\ &\leq \mathrm{C}\frac{\lambda^2}{n}\sum_{j=1}^{n_1}\frac{q_{(k-2)(n_1+n_2)+n_1}}{q_{(k-1)(n_1+n_2)+j}}((k-1)(n_1+n_2)+j)^\beta \\ &\leq \mathrm{C}\frac{\lambda^2}{n}n_1\mathrm{A}^{-n_2}n^\beta \leq \mathrm{C}\lambda^2 n_2^{5\beta}\mathrm{A}^{-n_2}, \end{aligned}$$

and similarly

$$|\int_0^1 \frac{\lambda^2}{2n}\mathrm{X}_{nk}^2 e^{i\frac{\lambda}{\sqrt{n}}(\mathrm{X}_{n1}+\cdots+\mathrm{X}_{n(k-1)})}\mathrm{d}\mu - \frac{\lambda^2}{2n}\int_0^1 \mathrm{X}_{nk}^2\mathrm{d}\mu.\int_0^1 e^{i\frac{\lambda}{\sqrt{n}}(\mathrm{X}_{n1}+\cdots+\mathrm{X}_{n(k-1)})}\mathrm{d}x| \leq \mathrm{C}\lambda^3 n_2^{5\beta}\mathrm{A}^{-n_2}.$$

On the other hand, by lemma 5, one has that

$$|\int_0^1 \frac{1}{2i}\int_0^{\frac{\lambda}{\sqrt{n}}\mathrm{X}_{nk}} e^{it}(\frac{\lambda}{\sqrt{n}}\mathrm{X}_{nk} - t)^2\mathrm{d}t.e^{i\frac{\lambda}{\sqrt{n}}(\mathrm{X}_{n1}+\cdots+\mathrm{X}_{n(k-1)})}\mathrm{d}\mu| \leq \mathrm{C}\frac{\lambda^3}{n^{3/2}}||\mathrm{X}_{nk}||_{\mathrm{L}^3}^3 \leq \mathrm{C}\frac{\lambda^3}{n^{3/2}}||\mathrm{X}_{nk}||_{\mathrm{L}^4}^3$$

$$\leq \mathrm{C}\frac{\lambda^3}{n^{3/2}}n_1^{3/2}\ln^{3/2} n \leq \mathrm{C}\lambda^3(\frac{1}{n^{1/4}})^{3/2}\ln^{3/2} n. \quad \square$$

We now fix $\lambda \in \mathbf{R}$. We define $\mathrm{J}_n(\lambda) = \int_0^1 e^{i\frac{\lambda}{\sqrt{n}}\mathrm{S}_n}\mathrm{d}\mu$. One has

$$|\mathrm{J}_n(\lambda) - \mathrm{I}_{np(n)}(\lambda)| \leq ||\frac{\lambda}{\sqrt{n}}\sum_{k=1}^{p(n)}\mathrm{Y}_{kn}||_{\mathrm{L}^\infty} \leq \mathrm{C}\lambda\frac{p(n)n_2}{\sqrt{n}} \leq \mathrm{C}\lambda n^{-1/20}. \tag{12}$$

For n large enough, one has $|1-(\lambda^2/2n)\int_0^1 \mathrm{X}_{nk}^2 \mathrm{d}x| \leq 1$. Thus, by (12) and by recursive application of lemma 6, one has (incorporating λ in the constant of the right hand side)

$$|\mathrm{J}_n(\lambda)-(1-\frac{\lambda^2}{2n}\int_0^1 \mathrm{X}_{np(n)}^2 \mathrm{d}\mu)\ldots(1-\frac{\lambda^2}{2n}\int_0^1 \mathrm{X}_{n1}^2 \mathrm{d}\mu)| \leq \mathrm{C}p(n)(n_2^{5\beta}\mathrm{A}^{-n_2}+(\frac{1}{n^{1/4}})^{3/2}\ln^{3/2} n)+\mathrm{C}n^{-1/20}.$$

The right hand side of this inequality goes to 0 as $n \to \infty$ (because $p(n)/n^{1/4} \to 1$).

To end the proof, one needs to show that $\ln \prod_{k=1}^{p(n)}(1-(\lambda^2/2n)\int_0^1 \mathrm{X}_{nk}\mathrm{d}\mu) \to -\lambda^2\sigma^2/2$ as $n \to \infty$. One has

$$\ln \prod_{k=1}^{p(n)}(1-\frac{\lambda^2}{2n}\int_0^1 \mathrm{X}_{nk}\mathrm{d}\mu) = -\frac{\lambda^2}{2n}\sum_{k=1}^{p(n)}\int_0^1 \mathrm{X}_{nk}^2 \mathrm{d}x\mu + O\frac{\lambda^4}{n^2}\sum_{k=1}^{p(n)}(\int_0^1 \mathrm{X}_{nk}^2\mathrm{d}\mu)^2 \quad \text{as} \quad n \to \infty.$$

By lemma 5 (and Hölder inequality), the rest term goes to 0 like $(\ln n)/n^{1/4}$ for $n \to \infty$. Proceeding as in the proof of lemma 6, one sees that $\int_0^1 \sum_{k=1}^{p(n)} \mathrm{X}_{nk}^2 \mathrm{d}\mu - \int_0^1 (\sum_{k=1}^{p(n)} \mathrm{X}_{nk})^2 \mathrm{d}\mu \to 0$ as $n \to \infty$. One concludes using (12) and hypothesis 3). □

Because hypothesis 3) of proposition 3 was used only at the very end of its proof, one has actually shown the following corollary.

Corollary 1 *Suppose that all the conditions of proposition 3 hold, except* 3). *Instead, suppose that there exist* $\mathrm{C}, \epsilon > 0$ *such that, for every* $n \geq 1$, $\epsilon \leq ||\mathrm{S}_n/\sqrt{n}||_{\mathrm{L}^2} \leq \mathrm{C}$. *Let* μ_n *be the distribution of* $\mathrm{S}_n/\sqrt{n}$, *and let* $\sigma_n = ||\mathrm{S}_n/\sqrt{n}||_{\mathrm{L}^2}$ $(n \geq 1)$. *Then* $\mu_n - g(\sigma_n) \rightharpoonup 0$.

4 Proof of Proposition 2

Let $\alpha \in \mathrm{E}(\mathrm{A}, d)$ for some $\mathrm{A} \geq 2$ and some $d \geq 1$. Let $(p_n/q_k)_{k\geq 0}$ be its convergents, and $(a_k)_{k\geq 0}$ its partial quotients. Let $\psi \in \mathrm{BV}(\mathbf{T},\mathbf{R})$ be such that $\int_0^1 \psi \mathrm{d}x = 0$. Let $f_1 = y_{q_1}$, and $f_k = \mathrm{R}_{r_{k-1}\alpha} y_{q_k}$ for $k \geq 2$. One has $y_{r_n} = \sum_{k=1}^n f_k$. Therefore, to prove proposition 2, it is enough to show that the sequence $(q_k)_{k\geq 1}$ and $(f_k)_{k\geq 1}$ satisfies the hypothesises of corollary 1 for $\mu = m_{\mathrm{L}}$. One has $\mathrm{Var}(f_n) \leq \mathrm{Var}(\psi) q_n$, and, by Denjoy-Koksma inequality, one has $||f_k||_{\mathrm{L}^\infty} \leq \mathrm{C}$ for all $k \geq 1$.

Therefore, one only needs to prove that $(f_n)_{n\geq 1}$ satisfies hypothesis 2) of proposition 3, and the specific hypothesis of corollary 1. Let us list four basic inequalities that are used repeatedly in the sequel (the first and the second where presented after (5), the third comes from the theory of continued fractions, the fourth comes from the fact that α is of constant type) :

$$\begin{aligned} &\forall x \in \mathbf{R}, 4|x|_{\mathbf{T}} \leq |1-e^{2i\pi x}| \leq 2\pi |x|_{\mathbf{T}}, \\ &\forall x \in \mathbf{R}, \forall m \in \mathbf{Z}, |1-e^{2i\pi m x}| \leq |m|.|1-e^{2i\pi x}|, \\ &\forall m \geq 1, |q_m \alpha|_{\mathbf{T}} \leq 1/q_{m+1} < 1/q_m, \\ &\exists \mathrm{C}(\alpha) > 0 : \forall k \in \mathbf{Z}_0, |k\alpha|_{\mathbf{T}} \geq \mathrm{C}(\alpha)/|k|. \end{aligned} \tag{13}$$

Lemma 7 *If* $\alpha \in \mathrm{E}(\mathrm{A}, d)$ *with* $\mathrm{A} \geq 2$, *then* $\forall p \in \mathbf{N}$, $\exists b_0 \in \mathbf{N} : (\forall b \in \mathbf{N} : b \geq b_0)$, $\forall n \in \mathbf{N}$, $q_{bn} \geq q_n^p$.

Proof. For $b \in \mathbf{N}$ large enough, one has $q_{bn} \geq 2^{bn} \geq (2d\mathrm{A})^{pn} \geq q_n^p$. $\square$

Lemma 8 *There exist* $\mathrm{C}, \beta \in \mathbf{R}$ *such that, for every* $\phi \in \mathrm{BV}(\mathbf{T}, \mathbf{R})$ *with* $\int_0^1 \phi \mathrm{d}x = 0$*, and for every* $t \geq s \geq 1$*,*

$$|\int_0^1 \phi f_s \mathrm{d}x| \leq \mathrm{C}.\mathrm{Var}(\phi)\frac{s^\beta}{q_s}, \qquad |\int_0^1 \phi f_s f_t \mathrm{d}x| \leq \mathrm{C}.\mathrm{Var}(\phi)\frac{s^\beta}{q_s}. \tag{14}$$

Proof. Both inequalities may be shown in the same way, but the first one is simpler, and we will only prove the second one. In this proof, we consider as constants, numbers that depend only on α or ψ. Let $t \geq s \geq 1$. One writes $g = f_s$ and $h = f_t$. Let $\phi \in \mathrm{BV}(\mathbf{T}, \mathbf{R})$. One has $|\hat{\phi}(k)| \leq \mathrm{Var}(\phi)/2\pi|k|$ if $k \in \mathbf{Z}_0$ and $\hat{\phi}(0) = 0$. Let us first simplify the problem in two ways.

First, by lemma 7, there exists $b \in \mathbf{N}$ such that $q_{bn} \geq q_n^4$ for every $n \in \mathbf{N}$. Let $\mathrm{P} : \mathrm{L}^2 \to \mathrm{L}^2$ be the projector definied by $\mathrm{P}u(x) = \sum_{k \leq q_{bt}} \hat{u}(k) e^{2i\pi kx}$. Let $\mathrm{Q} = \mathrm{Id} - \mathrm{P}$. One has $||\mathrm{Q}g||_{\mathrm{L}^2} \leq \mathrm{C}/q_t$ and $||\mathrm{Q}h||_{\mathrm{L}^2} \leq \mathrm{C}/q_t$. Indeed, one has for example

$$||\mathrm{Q}h||_{\mathrm{L}^2}^2 \leq \frac{\mathrm{Var}^2(h)}{4\pi^2} \sum_{k:|k|\geq q_{bt}} \frac{1}{k^2} \leq \mathrm{C}\frac{q_t^2}{q_{bt}} \leq \frac{\mathrm{C}}{q_t^2}.$$

But $|\int_0^1 \phi g h \mathrm{d}x| \leq |\int_0^1 \phi \mathrm{P}g\mathrm{P}h \mathrm{d}x| + \int_0^1 |\phi \mathrm{Q}g\mathrm{P}h| \mathrm{d}x + \int_0^1 |\phi g \mathrm{Q}h| \mathrm{d}x$. Therefore, because the two last terms of this sum are bounded by C/q_t, it will suffice to estimate the first one.

Next, let us prove that, if $t \geq cs$ for some $c \in \mathbf{N}$, then the first inequality (14) implies the second one. Indeed, $\int_0^1 \phi \mathrm{d}x = 0$, and so $||\phi||_{\mathrm{L}^\infty} \leq \mathrm{Var}(\phi)$. Therefore

$$|\int_0^1 \phi f_s f_t \mathrm{d}x| \leq \mathrm{C}(||f_s||_{\mathrm{L}^\infty}\mathrm{Var}(\phi) + \mathrm{Var}(\phi)\mathrm{Var}(f_s))\frac{t^\beta}{q_t} \leq \mathrm{C}\frac{\mathrm{Var}(\phi)}{q_s}(\frac{q_s t^\beta}{q_t} + \frac{q_s^2 t^\beta}{q_t}). \tag{15}$$

By lemma 7, there exists $c \in \mathbf{N}$ such that $q_{cn} \geq q_n^2$ for every $n \in \mathbf{N}$. therefore, if $t \geq cs$, one may write $t = cs + u$ with $u \geq 0$. One has therefore

$$\frac{q_s t^\beta}{q_t} + \frac{q_s^2 t^\beta}{q_t} \leq \frac{2q_{cs}(cs+u)^\beta}{q_{cs+u}} \leq \frac{2(cs+u)^\beta}{\mathrm{A}^u} \leq 2c^\beta s^\beta \frac{(1+u/cs)^\beta}{\mathrm{A}^u}.$$

One now comes to the proof itself (and one supposes $t < cs$). One has

$$\begin{aligned}
|\int_0^1 \phi \mathrm{P}g\mathrm{P}h \mathrm{d}x| &= |\sum_{j,k:|j|,|k|\leq q_{bt}} \hat{g}(j)\hat{h}(-k)\hat{\phi}(k-j)| \\
&\leq \frac{4\mathrm{Var}(\phi)}{2\pi} \sum_{1\leq j,k\leq q_{bt}, k\neq j} |\hat{g}(j)|.|\hat{h}(k)|.\frac{1}{|k-j|} + \frac{\mathrm{Var}(\phi)}{2\pi} \sum_{1\leq k\leq q_{bt}} \frac{|\hat{g}(k)|.|\hat{h}(k)|}{k} \\
&\triangleq \frac{\mathrm{Var}(\phi)}{2\pi}(4\mathrm{S}_1 + \mathrm{S}_2).
\end{aligned}$$

Let us now define the sets

$$\Gamma_1 = \{j \in \mathbf{N}_0 : 1/q_1 \leq |1 - e^{2i\pi j\alpha}|\}, \qquad \Gamma_n = \{j \in \mathbf{N}_0 : 1/q_n \leq |1 - e^{2i\pi j\alpha}| < 1/q_{n-1}\} \quad (n \geq 2).$$

One has $\mathbf{N}_0 = \bigcup_{n\geq 1} \Gamma_n$ and $\Gamma_m \cap \Gamma_n = \oslash$ if $m \neq n$. Moreover, by (13) and because $q_{m-1} \geq q_m/2d\mathrm{A}$ $(m \geq 1)$, there exists $\mathrm{C} > 0$ such that, for every $m \geq 1$,

$$j \in \Gamma_m \Rightarrow j \geq \mathrm{C}q_m \quad \text{and} \quad j,k \in \bigcup_{n\geq m} \Gamma_n \Rightarrow |k-j| \geq \mathrm{C}q_m. \tag{16}$$

Let us estimate S_1. By (16), there exists a constant $l \geq 0$ such that, if $j \in \Gamma_{m+l}$, then $j > q_m$ $(m \geq 1)$, and therefore

$$\mathrm{S}_1 = \sum_{1\leq m,n\leq bt+l} \sum_{\substack{j\in\Gamma_m, j\leq q_{bt} \\ k\in\Gamma_n, k\leq q_{bt}, k\neq j}} |\hat{g}(j)|.|\hat{h}(k)|.\frac{1}{|k-j|} \triangleq \sum_{1\leq m,n\leq bt+l} \mathrm{S}(m,n).$$

Let us fix $m,n \in \{1,\ldots,bt+l\}$ and estimate $\mathrm{S}(m,n)$. Let us first consider the case $m \leq n$. By (5) and (13), one has $|\hat{g}(j)| \leq \mathrm{C}q_m/q_s$ and $|\hat{h}(k)| \leq \mathrm{C}q_n/k$. Therefore, by (16), one has

$$\mathrm{S}(m,n) \leq \mathrm{C}\frac{q_m q_n}{q_s} \sum_{\substack{j\in\Gamma_m, j\leq_{bt} \\ k\in\Gamma_n, k\leq q_{bt}, k\neq j}} \frac{1}{k|k-j|} \leq \mathrm{C}\frac{q_m q_n}{q_s} \sum_{k\in\Gamma_n, k\leq q_{bt}} \frac{1}{k} \sum_{j\in\Gamma_m, j\leq q_{bt}, j\neq k} \frac{1}{|k-j|} \leq \frac{\mathrm{C}}{q_s} \ln^2 q_{bt}.$$

The case $m \geq n$ is analogous : one uses the estimates $|\hat{g}(j)| \leq \mathrm{C}q_m/j$ and $|\hat{h}(k)| \leq q_n/q_t$, to obtain $\mathrm{S}(m,n) \leq (\mathrm{C}/q_t)\ln^2 q_{bt}$. Therefore, one has $\mathrm{S}_1 \leq \mathrm{C}(bt+l)^2(\ln^2 q_{bt})/q_s$.

The sum S_2 is estimated in the same way. One gets $\mathrm{S}_2 \leq \mathrm{C}(bt+l)(\ln q_{bt})/q_s$. To get the result, one uses then the inequality $q_{bt} \leq (2d\mathrm{A})^{bt}$. $\square$

Lemma 9 *Let α be a number of constant type. Let $n \geq 0$ and $0 \leq m \leq q_n$. One has $||y_m||_{\mathrm{L}^2} \leq \mathrm{C}\sqrt{n}$.*

Proof. Let us retake the notations of the proof of lemma 8. One said there that there exists $c \in \mathbf{N}$ such that $q_{cn} \geq q_n^2$ for every $n \in \mathbf{N}$. By (5), one has

$$\int_0^1 y_m^2 \mathrm{d}x \leq \frac{\mathrm{Var}^2(\psi)}{2\pi^2} \sum_{k=1}^{q_{cn}-1} \frac{1}{k^2} \frac{|1-e^{2i\pi mk\alpha}|^2}{|1-e^{2i\pi k\alpha}|^2} + \frac{1}{2\pi^2} \sum_{k\geq q_{cn}} \frac{\mathrm{Var}^2(y_m)}{k^2}.$$

Because $\mathrm{Var}(y_m) \leq \mathrm{Var}(\psi)m \leq \mathrm{Var}(\psi)q_n$, the second term is bounded by a constant. But, proceeding as in the proof of lemma 8, one gets

$$\sum_{k=1}^{q_{cn}-1} \frac{1}{k^2} \frac{|1-e^{2i\pi mk\alpha}|^2}{|1-e^{2i\pi k\alpha}|^2} \leq \sum_{n=1}^{cn+l} \sum_{k\in\Gamma_n} \frac{1}{k^2} \frac{2}{|1-e^{2i\pi k\alpha}|^2} \leq \mathrm{C} \sum_{n=1}^{cn+l} q_n^2 \frac{1}{q_n^2} \sum_{k\geq 1} \frac{1}{k^2} \leq \mathrm{C}n. \quad \square$$

If $\alpha \in \mathrm{E}(\mathrm{A}, d)$, with $\mathrm{A} \geq 2$, then $r_n \leq q_{n+1}$ $(n \geq 1)$. Therefore, it follows from lemma 9 that $\int_0^1 (\sum_{k=1}^n f_k)^2 \mathrm{d}x = \int_0^1 y_{r_n}^2 \mathrm{d}x \leq \mathrm{C}n$ $(n \geq 1)$.

Lemma 10 *Let $\psi = \psi_*$. There exist $\epsilon > 0$, $\mathrm{A}_0 \geq 2$ such that, if $\mathrm{A} \geq \mathrm{A}_0$ and if $\alpha \in \mathrm{C}(\mathrm{A}, \nu)$, then, for every $n \geq 1$, $\int_0^1 (\sum_{k=1}^n f_k)^2 \mathrm{d}x \geq n\epsilon$.*

Proof. For $k \geq 1$, we define $\delta(k) = 0$ if k is even, and $\delta(k) = 1$ if k is odd. By 5, one has

$$\begin{aligned}\int_0^1 (\sum_{k=1}^n f_k)^2 \mathrm{d}x &= \int_0^1 y_{r_n}^2 \mathrm{d}x = \frac{8}{\pi^2} \sum_{k\geq 1} \frac{\delta(k)}{k^2} \frac{|1-e^{2i\pi r_n k\alpha}|^2}{|1-e^{2i\pi k\alpha}|^2} \geq \frac{8}{\pi^2} \sum_{s\geq 1} \frac{\delta(q_s)}{q_s^2} \frac{|1-e^{2i\pi r_n q_s\alpha}|^2}{|1-e^{2i\pi q_s\alpha}|^2} \\ &\geq \mathrm{C} \sum_{s=1}^n \delta(q_s) |1-e^{2i\pi r_n q_s\alpha}|^2 \geq \mathrm{C} \sum_{s=1}^n \delta(q_s) |r_n q_s \alpha|_{\mathbf{T}}^2.\end{aligned}$$

Let us fix $s \in \{1, \ldots, n\}$. We define $\tau = (q_1 + \cdots + q_{s-1})q_s\alpha + q_s(q_{s+1}\alpha + \cdots + q_n\alpha)$ $(1 \leq s \leq n)$. One has $r_n q_s \alpha = q_s q_s \alpha + \tau$. The inequality $|x+y|_{\mathbf{T}} \leq |x|_{\mathbf{T}} + |y|_{\mathbf{T}}$ is valid for any $x, y \in \mathbf{R}$, and so $|\tau|_{\mathbf{T}} \leq |q_1 q_s \alpha|_{\mathbf{T}} + \cdots + |q_{s-1} q_s \alpha|_{\mathbf{T}} + |q_s q_{s+1}\alpha|_{\mathbf{T}} + \cdots + |q_s q_n \alpha|_{\mathbf{T}}$. Moreover, if $p \in \mathbf{N}$, if $x \in \mathbf{R}$ and if $p|x|_{\mathbf{T}} \leq 1/2$, then $|px|_{\mathbf{T}} = p|x|_{\mathbf{T}}$. But one has

$$q_k |q_l \alpha|_{\mathbf{T}} \leq \frac{q_k}{q_{l+1}} \leq \frac{1}{\mathrm{A}} \frac{q_k}{q_l}. \tag{17}$$

Therefore, because $\mathrm{A}_0 \geq 2$,

$$|\tau|_{\mathrm{T}} \leq \frac{q_1}{q_{s+1}} + \cdots + \frac{q_{s-1}}{q_{s+1}} + \frac{q_s}{q_{s+2}} + \cdots + \frac{q_s}{q_{n+1}} \leq 2 \sum_{n=2}^{\infty} \frac{1}{\mathrm{A}^n}. \tag{18}$$

Next, the inequality $|x+y|_{\mathbf{T}} \geq |x|_{\mathbf{T}} - |y|_{\mathbf{T}}$ holds for each $x, y \in \mathbf{R}$ such that $|x|_{\mathbf{T}} + |y|_{\mathbf{T}} \leq 1/2$. Therefore, by (17) and (18), $|r_n q_s \alpha| \geq q_s |q_s \alpha|_{\mathbf{T}} - |\tau|_{\mathbf{T}}$ if A_0 is large enough. The inequality $|q_s \alpha|_{\mathbf{T}} \geq 1/(q_s + q_{s+1})$ implies $q_s |q_s \alpha|_{\mathbf{T}} \geq 1/3\nu\mathrm{A}$. Therefore, for A_0 large enough, one has

$$|r_n q_s \alpha|_{\mathbf{T}} \geq \frac{1}{3\nu\mathrm{A}} (1 - 6\nu \sum_{n\geq 1} \frac{1}{\mathrm{A}^n}) \geq \frac{1}{6\nu\mathrm{A}_0}.$$

This completes the proof, because, by lemma 1, if q_n is even, q_{n+1} is odd $(n \geq 1)$. □

Aknowledgements. I am very grateful to Professors J. Bricmont and A. Kupiainen for uselfuls discussions and comments.

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